Smoothing
(Green/Blue)

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6/24/2011

Many thanks to previous handouts by Kiran and Po.

1 Jensen's Inequality

A *convex* function is a function $f(t)$ for which the second derivative is nonnegative. This is, for most purposes, *equivalent* to having the property that for any a, b in the domain, $f(\frac{a+b}{2}) \leq \frac{f(a)+f(b)}{2}$. Then given positive weights $\lambda_1, \ldots, \lambda_n$ that sum to 1, Jensen's inequality says that

$$f(\lambda_1 x_1 + \ldots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \ldots + \lambda_n f(x_n).$$

If your function is concave (the opposite of convex), apply Jensen to $-f$. If you've got something where f looks like a product rather than a sum, take the logarithm and hope that the logarithm of components of f are convex or concave. If f is convex or concave on only a portion of the domain? Do it anyway, just be more careful.

In general, "smoothing" is the process of nudging around parameters of an inequality so that it becomes tighter. Or, even more generally, nudging things in a small (i.e. simple) way, and *watching closely*. Jensen's is one particular form of this, where all the variables are nudged to their mean simultaneously, which might solve the problem in one shot. But there are many more specific "nudges" to try if straight-up Jensen does not quite work. Try smoothing one pair of variables at a time, perhaps moving both variables by the same amount but in opposite directions, perhaps moving until one of them hits some crucial value, like the arithmetic mean of all the variables, or the max of all the variables; maybe consider a pair that includes the biggest or the smallest variable. In general, look for changes that "make progress" in some way, but are also "simple" in some way. And then *watch closely*.

2 Problems

1. [Rearrangement inequality] Show that the series $a_1 b_1 + \ldots + a_n b_n$ is maximized when the a's and b's are both sorted in the same direction, and minimized when they are sorted in opposite directions.

2. [AM-GM inequality] For $x_1, x_2, \ldots, x_n \geq 0$ show that

$$\sqrt[n]{x_1 x_2 \cdot \ldots \cdot x_n} \leq \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

3. [USAMO'74] For $a, b, c > 0$ prove $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$

4. [USAMO'99] Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \ldots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \ldots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

5. [Zvezda] Prove for all nonnegative numbers a, b, c:

$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$

6. [MOP'97] Given a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n > 0$ for all $n \geq 0$, such that the sequence $\{a^n x_n\}_{n=0}^{\infty}$ is convex for all $a > 0$, show that the sequence $\{\log x_n\}_{n=0}^{\infty}$ is also convex.

7. [India, '95] Let $x_1, \ldots, x_n$ be positive numbers whose sum is 1. Prove that

$$\frac{x_1}{\sqrt{1-x_1}} + \ldots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$

8. [Friendship Competition'88] For $a, b, c > 0$:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$$

9. Let A, B, C be the angles of a triangle. Prove that

   - $\sin A + \sin B + \sin C \leq 3\sqrt{3}/2$;
   - $\cos A + \cos B + \cos C \leq 3/2$;
   - $\sin A/2 \sin B/2 \sin C/2 \leq 1/8$;
   - $\cot A + \cot B + \cot C \geq \sqrt{3}$;

   (Not everything here is convex everywhere!)

10. [Vietnam, '98] Let $x_1, \ldots, x_n$ $(n \geq 2)$ be positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \cdots \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{n-1} \geq 1998.$$

(Beware of non-convexity.)

11. [USAMO'98] Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \ldots + \tan(a_n - \pi/4) \geq n - 1$$

Prove that $\tan a_0 \tan a_1 \ldots \tan a_n \geq n^{n+1}$.

12. [Arbelos] Let $a_1, a_2, \ldots$ be a convex sequence of real numbers. Show that for all $n \geq 1$,

$$\frac{a_1 + a_3 + \ldots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \ldots + a_{2n}}{n}.$$

13. [USAMO'93] Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \ldots$ – that is, $\{\log a_i\}$ is concave. Show that for each $n \geq 1$,

$$\frac{a_0 + \ldots + a_n}{n+1} \cdot \frac{a_1 + \ldots + a_{n-1}}{n-1} \geq \frac{a_0 + \ldots + a_{n-1}}{n} \cdot \frac{a_1 + \ldots + a_n}{n}.$$